(d) State D'Alembert's Ratio test for a series. Find if the series,

$$\frac{1}{2}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+\cdots \text{ is convergent.}$$

[This question paper contains 8 printed pages.]

13.01.2025(M)

Your Roll No..............

Sr. No. of Question Paper : 1382 I

Unique Paper Code : 2352011102

Name of the Paper : Elementary Real Analysis

Name of the Course : **B.Sc. (H) Mathematics (NEP-UGCF 2022)**

Semester : I – DSC-2

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. Part of the questions to be attempted together.
4. **All** questions carry equal marks.
5. Use of Calculator is not allowed.

1. (a) If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

(b) State the order properties of $\mathbb{R}$. Using it prove that if a, b, c are real numbers such that $a > b$, then $a + c > b + c$.

(c) Find all values of x satisfying $|x - 2| \leq x + 1$.

(d) Write the definition of Supremum and Infimum of a set. Give an example of a set having supremum and infimum, where the set

(i) contains its supremum and infimum

(ii) does not contain its supremum and infimum

2. (a) State and prove Archimedean property.

(b) Let 5 be a non-empty subset of $\mathbb{R}$ and $a > 0$, then show that

$$\sup(aS) = a \sup S$$

6. (a) State the Alternating Series test. Show that the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is convergent.

(b) Test the convergence of the series

$$\frac{1}{e} + \frac{4}{e^2} + \frac{27}{e^3} + \frac{256}{e^4} + \frac{3,125}{e^5} + \ldots.$$

(c) Define a conditionally convergent series and an absolutely convergent series. Test the series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^{3/2}}$ for absolute or conditional convergence.

5. (a) State and prove Cauchy Criterion for convergence of a series $\sum_{n=1}^{\infty} a_n$.

(b) Test the convergence of the following series :

(i) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

(ii) $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

(c) Prove that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, $p > 0$ is convergent for $p > 1$ and divergent for $p \leq 1$.

(d) Show that if the series Σu_n converges, then $\lim_{n\to\infty} u_n = 0$. Is the converse true? Justify your answer.

(c) Let (x_n) be a sequence in $\mathbb{R}$ and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim_{n\to\infty}(a_n) = 0$ and for some constant $K > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq Ka_n$ for all $n \geq m$, then prove that $\lim_{n\to\infty}(x_n) = x$.

(d) Using the definition of limit, show that

$$\lim_{n\to\infty}\left(\frac{4n+5}{3n+4}\right) = \frac{4}{3}.$$

3. (a) Let (x_n) and (y_n) be sequences of real number such that $\lim_{n\to\infty}(x_n) = x$ and $\lim_{n\to\infty}(y_n) = y$, then show that $\lim_{n\to\infty}(x_n + y_n) = x + y$.

(b) Let (x_n) be a sequence of positive real numbers such that $L = \lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right)$ exists. Show that if $L < 1$, then (x_n) converges and $\lim_{n\to\infty}(x_n) = 0$.

(c) State Squeeze theorem and show that if $z_n = \left(2^n + 3^n\right)^{\frac{1}{n}}$ then $\lim_{n\to\infty} z_n = 3$.

(d) Let $X = (x_n)$ be a sequence of real numbers defined by $x_1 = 1$ and

$$x_{n+1} = \sqrt{2 + x_n} \quad \text{for } n \in \mathbb{R}.$$

Show that the sequence (x_n) is convergent and find its limit.

4. (a) Prove that if a sequence (x_n) is a monotone decreasing and bounded below sequence of real numbers, then it is convergent.

(b) State Bolzano Weierstrass Theorem for Sequences. Show that the sequence $((-1)^n)$ is divergent.

(c) Find limit inferior and limit superior of the following sequences:

(i) $\left(\sin\left(\frac{n\pi}{4}\right)\right)$

(ii) $(3 + (-1)^n)$

(d) Show that every Cauchy sequence of real numbers is bounded. Is the converse true? Justify your answer.